

Stochastic epidemic SEIRS models with a constant latency period

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Abstract

In this paper we consider the stability of a class of deterministic and stochastic SEIRS epidemic models with delay. Indeed, we assume that the transmission rate could be stochastic and the presence of a latency period of r consecutive days, where r is a fixed positive integer, in the “exposed” individuals class E. Studying the eigenvalues of the linearized system, we obtain conditions for the stability of the free disease equilibrium, in both the cases of the deterministic model with and without delay. In this latter case, we also get conditions for the stability of the coexistence equilibrium. In the stochastic case we are able to derive a concentration result for the random fluctuations and then, using the Lyapunov method, that under suitable assumptions the free disease equilibrium is still stable.

Keywords: SEIRS model, stochastic delay differential equations, stability, perturbations

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Running head: A SEIRS stochastic model

1 Introduction

The mathematical models developed to describe the spread of a communicable disease are both deterministic and stochastic and they may involve many factors such as infectious agents, mode of transmission, incubation periods, infectious periods, susceptibility, etc...

A well known deterministic model in a closed population consisting of susceptible (S), infective (I) and recover (R) were considered by Kermack and McKendrick in [10]. Since then various epidemic deterministic models have been developed, such as SIR, SIS, SEIR and SEIRS models with or without a time delay (see e.g McCluskey [12], [13] and Huang et al. [9]). Here, the class E denotes individuals “exposed” to the disease, but not yet infectious.

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In this paper we assume that our epidemic model encompasses the class E. This is the case of several diseases like chickenpox (discrete-time version of this model has been considered in [4]). Contrary to most of the models defined in the literature, we consider a constant holding time in the E class. i.e., we will assume a latency period of r consecutive days, where r is a fixed positive integer. On the contrary, the permanence in the other classes is defined in a classical manner. Finally, we assume that recovered individuals may become susceptible again and for this reason our model will be a SEIRS model with delay.

More precisely, we will assume that the total size of the population will be fixed and equal to n and that all the individuals belong to one of the four classes S, E, I or R, where S denotes a susceptible individual, E an infected but not infectious individual (latency period), I an infected and infectious individual and R a recovered individual. These assumptions lead to the following set of ordinary differential equations, where $S(t)$ denotes the fraction of individuals that are susceptible and the same for $E(t)$, $I(t)$ and $R(t)$:

$$\begin{cases} dS(t) = -\beta S(t)I(t)dt + \gamma R(t)dt \\ dE(t) = \beta S(t)I(t)dt - \frac{1}{K_r}E(t-r)dt \\ dI(t) = \frac{1}{K_r}E(t-r)dt - \mu I(t)dt \\ dR(t) = \mu I(t)dt - \gamma R(t)dt \end{cases} \quad (1)$$

where β represent the disease transmission coefficient, μ the rate at which infectious individuals becomes recovered, γ the rate at which recovered individuals become again susceptible and K_r the rate of latency. Note that K_r is a parameter that depends on the delay r on E . The nonstandard equation in the model (1) is represented by the second one, since we assume that the change in the percentage of individuals in the class E depends on the difference between the number of individuals that enter into this class at time t and those that entered r units of time before.

Similar deterministic models have been considered in the literature. Huang *et al.* [9] consider the stability of SIR and SIRS models with constant time delay caused by latency in a host and expressed as a function of $S(t-r)$ and $I(t-r)$. In Huang *et al.* [8] the authors consider a SEIR model with constant latency time and infectious periods. Bai [1] considers a delayed SEIRS model with varying total population size where the delay is also expressed as a function of $S(t-r)$ and $I(t-r)$. Constant delays in epidemic models also appears in [3] to model relapse in infectious diseases.

We also consider a stochastic behaviour of the disease transmission, assuming that β may be random. To do this, we will assume that a infectious individual makes a random number

$$\beta dt + \varepsilon dW_t$$

of contacts with other individuals in a time interval $[t, t+dt)$, where $\{W_t, t \geq 0\}$ denotes a standard Brownian motion (see [6] p.879). Thus, the deterministic

SEIRS model (1) becomes stochastic in the following way

$$\begin{cases} dS(t) = [-\beta S(t)I(t) + \gamma R(t)]dt - \varepsilon S(t)I(t)dW_t \\ dE(t) = [\beta S(t)I(t) - \frac{1}{K_r}E(t-r)]dt + \varepsilon S(t)I(t)dW_t \\ dI(t) = [\frac{1}{K_r}E(t-r) - \mu I(t)]dt \\ dR(t) = [\mu I(t) - \gamma R(t)]dt \end{cases} \quad (2)$$

As far as we know, the literature of epidemic stochastic models is scarce. Nevertheless, we may cite Tornatore *et al* [15] who proposes a stochastic SIR model with distributed time delay and discuss its stability and Gray *et al.* [6] who presents the study of a SIS epidemic model.

Let us point that the presence of a constant delay leads to (stochastic) delay differential equations, which are not easy to handle mathematically. Usually in the literature, the constant time delay appears with some incidence rate, i.e., $F(S(t-r))G(I(t-r))$ or with a control factor $e^{-kr}S(t-r)I(t-r)$. These factors help to deal with the delay. On the contrary, in our paper, we deal with a fixed delay in the class E , that is, $E(t-r)$, since we consider that the number of individuals that pass from E to I at time t , depends only on the number of individuals at E at time $t-r$. We also have a coefficient K_r^{-1} that depends on r but the relation is not exponentially and it has been considered in order to discuss the validity and stability of the model depending on the value of r .

In Section 2 we study the deterministic model (1). We analyse first the case without delay $r = 0$ obtaining that when $\mu \geq \beta$ the free disease equilibrium is stable while when $\beta > \mu$ the stability holds in the point of coexistence equilibrium. Thus we can see that it does not depend on γ (notice that γ allows that some recovered individuals became again susceptibles). Then we study the delayed model. We get that to ensure the validity of the model it is necessary that $K_r \geq re$. Then, we show that when $\mu \geq \beta$ the free disease equilibrium remains asymptotically stable for any delay r .

Section 3 is devoted to deal with the stochastic model (2). Our aim is to study what happens with the stochastic fluctuations of the deterministic model. We show that the solutions of the perturbed system tends uniformly (and exponentially) to the solutions of the deterministic model when ε tends to zero. It holds for both models, with delay and without delay. Finally, we show that for the nondelayed model, under the condition

$$\mu - \beta - \frac{1}{\mu K_r} \frac{\varepsilon^2}{2} > 0.$$

the free disease equilibrium is asymptotically stable.

Finally, in the Appendix we recall some basic results about stability: methods based on the study of the roots of the associated characteristic functions (we use them in Section 2) and methods based on Lyauponov functionals (used in Section 3).

2 Deterministic model

Let us now consider the model (1) with the initial condition $E(s) = e_0, S(t) + I(t) + R(t) = 0$ for $s \in (-r, 0]$. In the first subsection we will deal with the model without delay, i.e. when $r = 0$. In the second subsection we will study what happens when we introduce the delay.

2.1 Analysis of the case without delay

2.1.1 Existence and Positivity of the Solution

Using standard method we get the existence and uniqueness of solution. It follows from the fact that if we start with $s_0 \geq 0, e_0 \geq 0, i_0 \geq 0$ and $r_0 \geq 0$, the region

$$\{(s, e, i, r) : s, e, i, r \geq 0; s + e + i + r \leq 1\}$$

is positively invariant.

For instance, if $S(t_1) = 0, E(t_1) > 0, I(t_1) > 0, R(t_1) > 0$ for some $t \geq 0$ then $\dot{S}(t_1) = \gamma R(t_1) > 0$ and there will exists ϵ such that $S(t) > 0$ for any $t \in (t_1, t_1 + \epsilon)$. All the other cases can be done by similar arguments. On the other hand, obviously $S(t) + E(t) + I(t) + R(t) \leq 1$.

2.1.2 Analysis of the equilibrium

To get equilibria, we have to compute the solution to the following equations:

$$\begin{cases} 0 = -\beta SI + \gamma R \\ 0 = \beta SI - \frac{1}{K_r} E \\ 0 = \frac{1}{K_r} E - \mu I \\ 0 = \mu I - \gamma R \end{cases} \quad (3)$$

with the restriction $S + E + I + R = 1$. We have that the free disease equilibrium $X^0 = (1, 0, 0, 0)$ exists for any values of the parameters. In the case $\mu \geq \beta$ one can check that no other equilibrium exists while in the case $\beta > \mu$ there exists also one point of coexistence equilibrium

$$X^* = \left(\frac{\mu}{\beta}, \frac{K_r(\beta - \mu)\mu\gamma}{\beta(\gamma K_r\mu + \gamma + \mu)}, \frac{\gamma(\beta - \mu)}{\beta(\gamma K_r\mu + \gamma + \mu)}, \frac{(\beta - \mu)\mu}{\beta(\gamma K_r\mu + \gamma + \mu)} \right)$$

Proposition 2.1 1. If $\mu \geq \beta$, then X^0 is asymptotically stable;

2. If $\mu < \beta$, then X^* exists and is asymptotically stable.

Remark 2.2 It can be checked easily that the basic reproduction number for this model is

$$R_0 = \frac{\beta}{\mu}.$$

So, the stability of the free disease equilibrium hold when $R_0 \leq 1$ while otherwise we have the stability of the coexistence equilibrium.

Proof of Proposition 2.1: Since we have the relation $S(t) = 1 - E(t) - I(t) - R(t)$, we can consider that we are dealing with the 3-dimensional system

$$\begin{cases} dE(t) = [\beta(1 - E(t) - I(t) - R(t))I(t) - \frac{1}{K_r}E(t-r)]dt \\ dI(t) = [\frac{1}{K_r}E(t-r) - \mu I(t)]dt \\ dR(t) = [\mu I(t) - \gamma R(t)]dt \end{cases} \quad (4)$$

The coefficient matrix of the linearized system at the free disease equilibrium is

$$\begin{pmatrix} -K_r^{-1} & \beta & 0 \\ K_r^{-1} & -\mu & 0 \\ 0 & \mu & -\gamma \end{pmatrix}$$

with eigenvalues

$$\begin{bmatrix} \frac{1}{2} \frac{-\mu K_r - 1 + \sqrt{\mu^2 K_r^2 - 2\mu K_r + 4K_r\beta + 1}}{K_r} \\ -\frac{1}{2} \frac{\mu K_r + 1 + \sqrt{\mu^2 K_r^2 - 2\mu K_r + 4K_r\beta + 1}}{K_r} \\ -\gamma \end{bmatrix}.$$

The last two are clearly negative. On the other hand, we have

$$\begin{aligned} & \frac{1}{2} \frac{-\mu K_r - 1 + \sqrt{\mu^2 K_r^2 - 2\mu K_r + 4K_r\beta + 1}}{K_r} \\ &= \frac{1}{2} \frac{-(\mu K_r + 1) + \sqrt{(\mu K_r + 1)^2 - 4K_r(\mu - \beta)}}{K_r} \end{aligned}$$

that is also negative when $\beta < \mu$. Thus, when $\beta < \mu$ all the eigenvalues are negative and so, the free disease equilibrium is locally asymptotically estable.

Let us consider now what happens around the coexistence equilibrium. The coefficient matrix of the linearized system at the coexistence equilibrium is now

$$A := \begin{pmatrix} -\frac{\beta\gamma K_r + \gamma + \mu}{K_r(\gamma K_r\mu + \gamma + \mu)} & \frac{\gamma K_r\mu^2 - \beta\gamma + 2\gamma\mu + \mu^2}{\gamma K_r\mu + \gamma + \mu} & -\frac{\gamma(\beta - \mu)}{\gamma K_r\mu + \gamma + \mu} \\ K_r^{-1} & -\mu & 0 \\ 0 & \mu & -\gamma \end{pmatrix}.$$

Our aim is to chek that when $\beta > \mu$ all the eigenvalues have negative real part. Using the well-known Routh-Hurwitz criterium, it is enough to chek that $\text{Trace}(A) < 0$, $\text{Determinant}(A) < 0$ and $-A_2 * \text{Trace}(A) + \text{Determinant}(A) > 0$ where A_2 is the coefficient of λ in the characteristic polynomial $P(\lambda)$ of A , i.e., if $A = (a_{i,j})$,

$$A_2 := a_{1,1}a_{2,2} + a_{1,1}a_{3,3} + a_{2,2}a_{3,3} - a_{1,2}a_{2,1} - a_{1,3}a_{3,1} - a_{2,3}a_{3,2}.$$

Indeed

$$\begin{aligned}\text{Trace}(A) &= -\frac{\beta \gamma K_r + \gamma + \mu}{K_r (\gamma K_r \mu + \gamma + \mu)} - \mu - \gamma < 0 \\ \text{Determinant}(A) &= -\frac{\gamma (\beta - \mu)}{K_r} < 0,\end{aligned}$$

and finally

$$A_2 = \frac{\gamma (\gamma K_r^2 \mu^2 + \beta \gamma K_r + \beta K_r \mu + \gamma K_r \mu + \beta + \gamma)}{K_r (\gamma K_r \mu + \gamma + \mu)}.$$

Notice that,

$$A_2 > \frac{\gamma (\beta K_r \mu + \beta)}{K_r (\gamma K_r \mu + \gamma + \mu)}, \quad -\text{Trace}(A) > \mu + \gamma,$$

and

$$\text{Determinant}(A) > -\frac{\gamma \beta}{K_r}.$$

So

$$\begin{aligned}& -A_2 * \text{Trace}(A) + \text{Determinant}(A) \\ & > \frac{\gamma (\beta K_r \mu + \beta)}{K_r (\gamma K_r \mu + \gamma + \mu)} (\mu + \gamma) - \frac{\gamma \beta}{K_r} = \frac{\gamma \beta K_r \mu^2}{K_r (\gamma K_r \mu + \gamma + \mu)} > 0.\end{aligned}$$

□

2.2 Analysis of the delayed case

2.2.1 Existence and Positivity of the solution

The system (1) can be solved step by step. Indeed, if we are able to solve the system up to time nr , we can find the solution for $t \in [nr, (n+1)r]$, since

$$I(t) = e^{-\mu(t-nr)} \left(\frac{1}{K_r} \int_{nr}^t E(s-r) e^{\mu(s-nr)} ds + I(nr) \right)$$

and moreover

$$\begin{aligned}R(t) &= e^{-\gamma(t-nr)} \left(\int_{nr}^t \mu I(s) e^{\gamma(s-nr)} ds + R(nr) \right) \\ S(t) &= e^{-\int_{nr}^t \beta I(s) ds} \left(\int_{nr}^t \gamma R(s) e^{\int_{nr}^s \beta I(u) du} ds + S(nr) \right) \\ E(t) &= E(nr) + \int_{nr}^t \beta S(s) I(s) ds - \frac{1}{K_r} \int_{nr}^t E(s-r) ds.\end{aligned}$$

On the other hand, we can reduce the problem to the study of the positivity of E , since if E is nonnegative on $[0, t]$ then I, R and S are clearly nonnegative

functions on $[0, t+r]$. Moreover, if I and S are nonnegative on $[0, t+r]$ we have that

$$\frac{dE}{ds} = \beta S(s)I(s) - \frac{1}{K_r}E(s-r) \geq -\frac{1}{K_r}E(s-r)$$

and using a comparison argument $E(s) \geq F(s)$ where F is the solution of the equation

$$\frac{dF}{ds} = -\frac{1}{K_r}F(s-r).$$

The next remark helps us to justify the positivity of the solutions.

Remark 2.3 *Let us consider the delay differential equation*

$$\frac{dF}{ds} = -KF(s-r)$$

on $[0, t]$ where K is a positive constant. From Theorem A in [14] it follows that if $K \leq \frac{1}{r}e^{-1}$ then the solution of this equation is nonnegative. Furthermore, Corollary 2.1 in [2] yields that if $K = \frac{1}{r}e^{-1}$ then the solution is nonnegative and $\lim_{s \rightarrow \infty} F(s) = 0$.

So, we can state the existence and positivity in the following proposition.

Proposition 2.4 *Assume that $\beta, \mu, \gamma \in (0, 1)$ and $K_r \geq re$, then the system (1) has an unique nonnegative solution.*

2.2.2 Analysis of the equilibrium points

The linearization of the delayed SEIR system about the steady state $(1, 0, 0, 0)$ is

$$\begin{pmatrix} \frac{dS}{dt} \\ \frac{dE}{dt} \\ \frac{dI}{dt} \\ \frac{dR}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\beta & +\gamma \\ 0 & 0 & \beta & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & \mu & -\gamma \end{pmatrix} \begin{pmatrix} S(t) \\ E(t) \\ I(t) \\ R(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -K_r^{-1} & 0 & 0 \\ 0 & K_r^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} S(t-r) \\ E(t-r) \\ I(t-r) \\ R(t-r) \end{pmatrix}.$$

As in the nondelayed case we can eliminate the first equation since $S(t) = 1 - (E(t) + I(t) + R(t))$. Thus we get the characteristic equation

$$P(\lambda) = (\lambda + \gamma) \left(\lambda^2 + \lambda\mu + \left(\lambda \frac{1}{K_r} + \left(\frac{\mu}{K_r} - \frac{\beta}{K_r} \right) \right) e^{-\lambda r} \right).$$

Let us fix K_r and let us study what happens when the delay r is increasing if $\beta < \mu$. We have to study the behaviour of the roots of the characteristic function. Clearly $-\gamma$ is a negative real root. So, our aim will be the study of the other factor

$$\lambda^2 + \lambda\mu + \left(\lambda \frac{1}{K_r} + \left(\frac{\mu}{K_r} - \frac{\beta}{K_r} \right) \right) e^{-\lambda r} = 0.$$

Applying Proposition 5.1 with $a_1 = \mu, a_0 = 0, b_1 = K_r^{-1}, b_0 = K_r^{-1}(\mu - \beta)$, it is easy to check that $a_0 + b_0 > 0, a_1 + b_1 > 0$, and $a_0^2 < b_0^2$ and consequently there will be positive roots for some r and the steady state $(1, 0, 0, 0)$ became unstable when r is increasing.

Following the results in Subsection 5.1 and applying (13), (14) and (15), the state $(1, 0, 0, 0)$ remains asymptotically stable until $r^* = \frac{\theta}{\omega}$, where $\omega > 0, 0 \leq \theta < 2\pi$ and

$$\begin{aligned}\omega^2 &= \frac{1}{2} \left((K_r^{-2} - \mu^2) + ((K_r^{-2} - \mu^2)^2 + 4K_r^{-2}(\mu - \beta)^2)^{\frac{1}{2}} \right), \\ \cos \theta &= -\frac{\mu K_r^{-1} \omega^2 - \omega^2(\mu - \beta) K_r^{-1}}{K_r^{-1} \omega^2 + K_r^{-2}(\mu - \beta)^2} = -\frac{\beta K_r^{-1} \omega^2}{K_r^{-1} \omega^2 + K_r^{-2}(\mu - \beta)^2}, \\ \sin \theta &= \frac{K_r^{-1}(\mu - \beta) \mu \omega + K_r^{-1} \omega^3}{K_r^{-1} \omega^2 + K_r^{-2}(\mu - \beta)^2}.\end{aligned}$$

Since $\beta < \mu$, we clearly have that $\cos \theta < 0$ and $\sin \theta > 0$ and furthermore that $\theta \geq \frac{\pi}{2}$. Thus, if

$$r \leq M(K_r, \mu, \beta) := \frac{\pi}{2^{\frac{1}{2}} \left((K_r^{-2} - \mu^2) + ((K_r^{-2} - \mu^2)^2 + 4K_r^{-2}(\mu - \beta)^2)^{\frac{1}{2}} \right)^{\frac{1}{2}}}.$$

the state $(1, 0, 0, 0)$ remains asymptotically stable. Notice that

$$\begin{aligned}M(K_r, \mu, \beta) &\geq M(K_r, \mu, 0) \\ &= \frac{\pi}{2^{\frac{1}{2}} \left((K_r^{-2} - \mu^2) + ((K_r^{-2} - \mu^2)^2 + 4K_r^{-2}\mu^2)^{\frac{1}{2}} \right)^{\frac{1}{2}}} \\ &= \frac{1}{2} \pi K_r.\end{aligned}$$

Thus, since $r < \frac{K_r}{e}$, it holds that $r < \frac{1}{2} \pi K_r \leq M(K_r, \mu, \beta)$ and the state $(1, 0, 0, 0)$ remains asymptotically stable for any possible delay. The result states as follows:

Proposition 2.5 *Assume that $\beta, \mu, \gamma \in (0, 1)$ and $K_r \geq re$, then the free disease equilibrium point is asymptotically stable for any $r > 0$.*

Let us study now what happens with the coexistence equilibrium when $\beta > \mu$. The linearization of the delayed SEIR system about the coexistence

equilibrium, reduced to three equations, is

$$\begin{pmatrix} \frac{dE}{dt} \\ \frac{dI}{dt} \\ \frac{dR}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{(\beta-\mu)\gamma}{\gamma K_r \mu + \gamma + \mu} & -\frac{(\beta-\mu)\gamma}{\gamma K_r \mu + \gamma + \mu} + \mu & -\frac{(\beta-\mu)\gamma}{\gamma K_r \mu + \gamma + \mu} \\ 0 & -\mu & 0 \\ 0 & \mu & -\gamma \end{pmatrix} \begin{pmatrix} E(t) \\ I(t) \\ R(t) \end{pmatrix} + \begin{pmatrix} -K_r^{-1} & 0 & 0 \\ K_r^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E(t-r) \\ I(t-r) \\ R(t-r) \end{pmatrix}.$$

As in the previous case we get the characteristic equation

$$\lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 + (b_2 \lambda^2 + b_1 \lambda + b_0) e^{-\lambda r} = 0. \quad (5)$$

with

$$\begin{aligned} a_0 &= \frac{\gamma^2 \mu (\beta - \mu)}{\gamma K_r \mu + \gamma + \mu}, \\ a_1 &= \frac{\gamma (\gamma K_r \mu^2 + \beta \gamma + \beta \mu)}{\gamma K_r \mu + \gamma + \mu}, \\ a_2 &= \frac{\gamma^2 K_r \mu + \gamma K_r \mu^2 + \beta \gamma + \gamma^2 + \gamma \mu + \mu^2}{\gamma K_r \mu + \gamma + \mu}, \\ b_0 &= \frac{\gamma (\beta \gamma + \beta \mu - \gamma \mu - \mu^2)}{(\gamma K_r \mu + \gamma + \mu) K_r}, \\ b_1 &= \frac{\gamma (\gamma K_r \mu + \beta + \gamma)}{(\gamma K_r \mu + \gamma + \mu) K_r}, \\ b_2 &= K_r^{-1}. \end{aligned}$$

Applying Proposition 5.2 to our characteristic function we get that the state will become unstable with increasing delay since it is easy to check that $a_2 + b_2 > 0$, $a_0 + b_0 > 0$, $(a_2 + b_2)(a_1 + b_1) - (a_0 + b_0) > 0$ and, with the additional condition $K_r < \frac{1}{\mu} + \frac{1}{\gamma}$, $C = a_0^2 - b_0^2 < 0$.

Although this case is more complicated and it is not possible to give a general criterium as in the endemic case, following the methods in Kuang in [11] pages 74-76, we will find some conditions about the first crossing of the imaginary axis. That is, we will find a point to which the steady state will remain asymptotically stable.

Let us assume that $\lambda = i\omega$, $\omega > 0$, is a root for some r . Under the hypothesis $a_0 + b_0 \neq 0$, we clearly have that $\omega \neq 0$. Then, we have

$$\begin{aligned} (a_0 - a_2 \omega^2) + b_1 \omega \sin(\omega r) + (b_0 - b_2 \omega^2) \cos(\omega r) &= 0 \\ (-\omega^3 + a_1 \omega) + b_1 \omega \cos(\omega r) + (b_0 - b_2 \omega^2) \sin(\omega r) &= 0. \end{aligned}$$

So

$$\begin{aligned} 0 &= (a_2^2\omega^2 - a_0)^2 + (\omega^3 - a_1\omega)^2 - b_1^2\omega^2 - (b_0 - b_0\omega^2)^2 \\ &= \omega^6 + A\omega^4 + B\omega^2 + C, \end{aligned} \quad (6)$$

where A, B and C are defined in (17). Let us consider now the associated discriminant to the third degree equation (6)

$$\Delta := 18ABC - 4A^3C + A^2B^2 - 4B^2 - 27C^2.$$

Then if $\Delta < 0$, the equation (6) (as an equation of ω^2) has only one real root ω_0 and the system will remain stable unless until the delay $r = \frac{\theta}{\omega_0}$ where

$$\begin{aligned} \cos \theta &= -\frac{(-\omega_0^3 + a_1\omega_0)b_1\omega_0 + (a_0 - a_2\omega_0^2)(b_0 - b_2\omega_0^2)}{b_1^2\omega_0^2 + (b_0 - b_2\omega_0^2)^2} \\ \sin \theta &= \frac{(b_0 - b_2\omega_0^2)(-\omega_0^3 + a_1\omega_0) - b_1\omega_0(a_0 - a_2\omega_0^2)}{b_1^2\omega_0^2 + (b_0 - b_2\omega_0^2)^2}. \end{aligned} \quad (7)$$

This leads to the following result:

Proposition 2.6 *Assume that $\beta, \mu, \gamma \in (0, 1)$ and $K_r \geq re$ with $K_r < \frac{1}{\mu} + \frac{1}{\gamma}$, then if $\Delta < 0$ the coexistence equilibrium point is asymptotically stable for any $r < \frac{\theta}{\omega_0}$ where θ and ω_0 satisfy (7).*

3 Stochastic model

In this section we get some stability properties for the stochastic system. More precisely we will study the fluctuations of the random system. Let us recall that we are considering the system:

$$\begin{cases} dS^\varepsilon(t) = (-\beta S^\varepsilon(t)I^\varepsilon(t) + \gamma R^\varepsilon(t))dt + \varepsilon S^\varepsilon(t)I^\varepsilon(t)dW(t) \\ dE^\varepsilon(t) = (\beta S^\varepsilon(t)I^\varepsilon(t) - \frac{1}{K_r}E^\varepsilon(t-r))dt - \varepsilon S^\varepsilon(t)I^\varepsilon(t)dW(t) \\ dI^\varepsilon(t) = (\frac{1}{K_r}E^\varepsilon(t-r) - \mu I^\varepsilon(t))dt \\ dR^\varepsilon(t) = (\mu I^\varepsilon(t) - \gamma R^\varepsilon(t))dt \end{cases}$$

We get a concentration result for the random fluctuations, that holds for the delayed and the non-delayed systems. This exponential stability states as follows.

Proposition 3.1 *Assume that $\beta, \mu, \gamma \in (0, 1)$ and $K_r \geq re$. Set $Z^\varepsilon(t)$ for the random vector $(S^\varepsilon(t), E^\varepsilon(t), I^\varepsilon(t), R^\varepsilon(t))$ and $Z(t)$ for the solution to the corresponding deterministic system. Then, there exists nonnegative constants K_1, K_2 depending on r, β, μ, γ such that*

$$P(\|Z^\varepsilon - Z\|_{\infty, [0, T]} > \rho) \leq \exp\left(-\frac{\rho^2}{\varepsilon^2 K_1 T \exp(K_2 T)}\right).$$

Proof: Set $J^\varepsilon(t) := \varepsilon \int_0^t S^\varepsilon(s) I^\varepsilon(s) dW(s)$. We can write, using that S^ε and I are bounded:

$$\begin{aligned}
|S^\varepsilon(t) - S(t)| &\leq \beta \int_0^t |S^\varepsilon(u) I^\varepsilon(u) - S(u) I(u)| du \\
&\quad + \int_0^t \gamma |R^\varepsilon(u) - R(u)| du + |J^\varepsilon(t)| \\
&\leq \beta \int_0^t |S^\varepsilon(u) - S(u)| du + \beta \int_0^t |I^\varepsilon(u) - I(u)| du \\
&\quad + \gamma \int_0^t |R^\varepsilon(u) - R(u)| du + |J^\varepsilon(t)|.
\end{aligned}$$

Analogously, using that $E^\varepsilon(u-r) - E(u-r) = 0$ for any $u \in (0, r)$, we get:

$$\begin{aligned}
|E^\varepsilon(t) - E(t)| &\leq \beta \int_0^t |S^\varepsilon(u) - S(u)| du + \beta \int_0^t |I^\varepsilon(u) - I(u)| du \\
&\quad + \int_0^t \frac{1}{K_r} |E^\varepsilon(u-r) - E(u-r)| du + |J^\varepsilon(t)| \\
&\leq \beta \int_0^t |S^\varepsilon(u) - S(u)| du + \beta \int_0^t |I^\varepsilon(u) - I(u)| du \\
&\quad + \frac{1}{K_r} \int_0^t |E^\varepsilon(u) - E(u)| du + |J^\varepsilon(t)|,
\end{aligned}$$

$$|I^\varepsilon(t) - I(t)| \leq \frac{1}{K_r} \int_0^t |E^\varepsilon(u) - E(u)| du + \mu \int_0^t |I^\varepsilon(u) - I(u)| du,$$

$$|R^\varepsilon(u) - R(u)| \leq \mu \int_0^t |I^\varepsilon(u) - I(u)| du + \gamma \int_0^t |R^\varepsilon(u) - R(u)| du.$$

Putting together these inequalities we obtain the existence of two positive constants K_1 and K_2 such that

$$|Z^\varepsilon(t) - Z(t)| \leq K_1 |J^\varepsilon(t)|^2 + K_2 \int_0^t |Z^\varepsilon(u) - Z(u)| du.$$

Applying classical Gronwall's lemma, we get for all $t \in [0, T]$:

$$|Z^\varepsilon(t) - Z(t)| \leq K_1 |J^\varepsilon(t)|^2 \exp(K_2 T).$$

So,

$$\begin{aligned}
P(\|Z^\varepsilon - Z\|_{\infty, [0, T]} > \rho) &\leq P\left(\|J^\varepsilon\|_{\infty, [0, T]}^2 > \frac{\rho^2}{K_1 \exp(K_2 T)}\right) \\
&= P\left(\sup_{t \in [0, T]} \left| \int_0^t S^\varepsilon(s) I^\varepsilon(s) dW(s) \right|^2 > \frac{\rho^2}{\varepsilon^2 K_1 \exp(K_2 T)}\right) \\
&\leq \exp\left(-\frac{\rho^2}{\varepsilon^2 K_1 T \exp(K_2 T)}\right),
\end{aligned}$$

where in the last inequality we have used the exponential martingale inequality and the fact that $\int_0^T (S^\varepsilon(s)I^\varepsilon(s))^2 ds \leq T$. \square

3.1 Analysis of the nondelayed system

Using Lyauponov functionals (see e.g. [9]), we can get a condition for ε such that the free disease equilibrium is asymptotically stable for the non delayed stochastic system.

Proposition 3.2 *Assume that $\beta, \mu, \gamma \in (0, 1)$; if*

$$\mu > \frac{\beta + \sqrt{\beta^2 + 2\varepsilon^2/K_r}}{2}$$

then the free disease equilibrium point is globally asymptotically stable for the non delayed stochastic system. Note that this condition implies that $\mu > \beta$.

Proof: We prove the stability of the disease-free equilibrium $E_0 = (1, 0, 0, 0)$. Using that $S(t) = 1 - E(t) - I(t) - R(t)$, we can consider that we have a system with three equations. Putting $u_1 = E, u_2 = I, u_3 = R$ we can consider the linearized system around $(0, 0, 0)$:

$$\begin{cases} du_1(t) = (\beta u_2(t) - \frac{1}{K_r} u_1(t))dt - \varepsilon u_2(t)dW(t) \\ du_2(t) = (\frac{1}{K_r} u_1(t) - \mu u_2(t))dt \\ du_3(t) = (\mu u_2(t) - \gamma u_3(t))dt \end{cases} \quad (8)$$

We denote $u = (u_1, u_2, u_3)$ and we consider the function

$$V(u) = u_1^2 + V_2 u_2^2 + V_3 u_3^2,$$

with $V_2, V_3 > 0$. Clearly $V \geq 0$ and $V(0, 0, 0) = 0$. We have

$$LV = 2(\beta + V_2 \frac{1}{K_r})u_1 u_2 + 2V_3 \mu u_2 u_3 - 2\frac{1}{K_r} u_1^2 - (2V_2 \mu - \varepsilon^2)u_2^2 - 2V_3 \gamma u_3^2.$$

To get that $LV \leq 0$ and using that $2u_1 u_2 \leq \lambda_1^2 u_1^2 + \frac{1}{\lambda_1^2} u_2^2$ and $2u_2 u_3 \leq \lambda_3^2 u_3^2 + \frac{1}{\lambda_3^2} u_2^2$, it is enough to impose that

$$\begin{aligned} -2\frac{1}{K_r} + \lambda_1^2(\beta + V_2 \frac{1}{K_r}) &\leq 0 \\ -2V_2 \mu + \varepsilon^2 + \frac{1}{\lambda_1^2}(\beta + V_2 \frac{1}{K_r}) + \frac{1}{\lambda_3^2} V_3 \mu &\leq 0 \\ -2V_3 \gamma + \lambda_3^2 V_3 \mu &\leq 0 \end{aligned}$$

Choosing $\lambda_1^2 = \frac{2\frac{1}{K_r} - \alpha_0}{\beta + V_2 \frac{1}{K_r}} > 0$ with $\alpha_0 > 0$, $\lambda_3^2 = \frac{\gamma}{\mu}$ and V_3 small enough, choosing α_0 as small as we want it suffices to get that

$$-2V_2 \mu + \varepsilon^2 + \frac{1}{2\frac{1}{K_r}}(\beta + V_2 \frac{1}{K_r})^2 < 0$$

i.e.

$$\frac{1}{K_r^2}V_2^2 + 2\frac{1}{K_r}(\beta - 2\mu)V_2 + \beta^2 + 2\frac{1}{K_r}\varepsilon^2 < 0. \quad (9)$$

Assuming that $2\mu - \beta > 0$, the minimum will be at the point $V_2 = K_r(2\mu - \beta)$. Furthermore, if

$$\mu - \beta - \frac{1}{\mu K_r} \frac{\varepsilon^2}{2} > 0. \quad (10)$$

which is equivalent, for positive μ , to be

$$\mu > \frac{\beta + \sqrt{\beta^2 + 2\varepsilon^2/K_r}}{2},$$

then Inequality (9) holds. Thus, the proof finishes applying Theorem 5.3 with

$$\begin{aligned} a(|u|) &:= \min(1, V_2, V_3)|u|^2, \\ b(|u|) &:= \max(1, V_2, V_3)|u|^2, \\ c(|u|) &:= \min\left(2\frac{1}{K_r} - \lambda_1^2(\beta + V_2\frac{1}{K_r}), 2V_2\mu - \varepsilon^2 - \frac{1}{\lambda_1^2}(\beta + V_2\frac{1}{K_r}) - \frac{1}{\lambda_3^2}V_3\mu, \right. \\ &\quad \left. 2V_3\gamma - \lambda_3^2V_3\mu\right)|u|^2. \end{aligned}$$

□

4 Conclusions and future work

In this paper we have analysed the stability of the equilibrium points of a family of SEIRS models. We consider both deterministic and stochastic models with or without delay proving that the free disease equilibrium is, under suitable assumptions, always asymptotic stable and that a similar result for the coexistence equilibrium only holds in some cases.

As a future work we plan to extend these results to models where different time delays are present in all the equations, to better describe the epidemic models. For example, if we consider the following general model, for $t > r_{EI}$,

$$\begin{cases} dS(t) = -\beta S(t)I(t)dt + \gamma R(t)dt \\ dE(t) = \beta S(t)I(t)dt - \beta S(t - r_E)I(t - r_E)dt \\ dI(t) = \beta S(t - r_E)I(t - r_E)dt - \beta S(t - r_{EI})I(t - r_{EI})dt \\ dR(t) = \beta S(t - r_{EI})I(t - r_{EI})dt - \gamma R(t)dt \end{cases} \quad (11)$$

where we assume that any individual remains in the classes E and I, respectively, for a constant amount of time equal to r_E and r_I and $r_{EI} = r_E + r_I$.

It is easy to see that the possible stability points for such a model are $(1, 0, 0, 0)$ when $r_I < \frac{1}{\beta}$ and $(1, 0, 0, 0)$ and

$$\left(\frac{1}{\beta r_I}, \frac{r_E}{r_I} \frac{\gamma(\beta r_I - 1)}{\beta(\gamma r_{EI} + 1)}, \frac{\gamma(\beta r_I - 1)}{\beta(\gamma r_{EI} + 1)}, \frac{1}{r_I} \frac{(\beta r_I - 1)}{\beta(\gamma r_{EI} + 1)} \right)$$

for $r_I \geq \frac{1}{\beta}$. In a forthcoming paper we will deal with the study of the stability of the previous equilibrium points, since the techniques applied in this paper appears not adequate.

5 Appendix

In this appendix we recall some well-known results about stability. First we deal with the study of the characteristic roots for deterministic delayed models. Finally we give some results about stochastic stability using Lyapunov functionals.

A stable steady state in a deterministic model can become unstable if, by increasing the delay, a characteristic root changes from having a negative real part to having positive real part. We will recall here some results about characteristic functions of order two and three.

5.1 Deterministic case: the degree two equation

Consider the characteristic function of degree two associated to a delayed system

$$\lambda^2 + a_1\lambda + a_0 + (b_1\lambda + b_0)e^{-\lambda r} = 0. \quad (12)$$

A steady state in this case is stable for $r = 0$ if all the roots of

$$\lambda^2 + (a_1 + b_1)\lambda + (a_0 + b_0) = 0$$

have negative real part. This occurs if and only if $a_1 + b_1 > 0$ and $a_0 + b_0 > 0$ (by Routh-Hurwitz conditions). We recall a result about the delayed system (Proposition 2.3 in [5]):

Proposition 5.1 *A steady state with characteristic equation (12) is stable in the absence of delay, and becomes unstable with increasing delay if and only if*

1. $a_0 + b_0 > 0$ and $a_1 + b_1 > 0$, and
2. $a_0^2 < b_0^2$, or $a_0^2 > b_0^2$, $a_1^2 < b_1^2 + 2a_0$ and $(a_1^2 - b_1^2 - 2a_0)^2 > 4(a_0^2 - b_0^2)$.

Moreover, as a particular case of the results in Kuang in [11] page 74-76, we have that under the hypothesis $a_0 + b_0 \neq 0$ and $a_0^2 < b_0^2$, the characteristic function (12) has only one imaginary solution $\lambda = i\omega, \omega > 0$

$$\omega^2 = \frac{1}{2} \left((b_1^2 + 2a_0 - a_1^2) + ((b_1^2 + 2a_0 - a_1^2)^2 - 4(a_0^2 - b_0^2))^{\frac{1}{2}} \right). \quad (13)$$

Then, the only crossing of the imaginary axis is from left to right as the delay increases. So the stability can only be lost and not regained. Furthermore the steady state remains asymptotically stable until $r^* = \frac{\theta}{\omega}$, where $0 \leq \theta < 2\pi$ with

$$\cos \theta = -\frac{a_1 b_1 \omega^2 + (a_0 - \omega^2) b_0}{b_1 \omega^2 + b_0^2} \quad (14)$$

$$\sin \theta = \frac{a_1 b_0 \omega - (a_0 - \omega^2) b_1 \omega}{b_1 \omega^2 + b_0^2}. \quad (15)$$

5.2 Deterministic case: the degree three equation

We consider now a three degree general characteristic equation

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda r} = 0. \quad (16)$$

As in the degree two case, a steady state is stable for $r = 0$ if all the roots of

$$\lambda^3 + (a_2 + b_2)\lambda^2 + (a_1 + b_1)\lambda + (a_0 + b_0) = 0$$

have negative real part. This occurs if and only if $a_2 + b_2 > 0$, $a_0 + b_0 > 0$ and $(a_2 + b_2)(a_1 + b_1) - (a_0 + b_0) > 0$. Moreover, we will recall a result about the delayed system (Proposition 2.4 in [5]). Set:

$$A := a_2^2 - b_2^2 - 2a_1, B := a_1^2 - b_1^2 + 2b_2b_0 - 2a_2a_0 \text{ and } C := a_0^2 - b_0^2. \quad (17)$$

Then:

Proposition 5.2 *A steady state with characteristic equation (16) is stable in the absence of delay, and becomes unstable with increasing delay if and only if A, B and C are not all positive and*

1. $a_2 + b_2 > 0$, $a_0 + b_0 > 0$ and $(a_2 + b_2)(a_1 + b_1) - (a_0 + b_0) > 0$, and
2. either $C < 0$, or $C > 0$, $A^2 - 3B > 0$ and $4(B^2 - 3AC)(A^2 - 3B) - (9C - AB)^2 > 0$.

5.3 Stochastic stability

Consider the n dimensional stochastic system

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW_t$$

where $f(t, x)$ is a function in \mathbb{R}^n defined in $[t_0, +\infty) \times \mathbb{R}^n$, and $g(t, x)$ is a $n \times n$ matrix, f, g are Locally Lipschitz functions in x and W is an m -dimensional Wiener process.

Let us denote by L the associated differential operator, defined for a non-negative function $V(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ by

$$LV = \frac{\partial V}{\partial t} + f^T \cdot \frac{\partial V}{\partial x} + \frac{1}{2} \text{Tr} \left[g^T \cdot \frac{\partial^2 V}{\partial x^2} \cdot g \right].$$

Recall that V is called a Lyapunov functional. The result about stability states as follows:

Theorem 5.3 *Suppose that there exist a non-negative function $V(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$, two continuous function $a, b : \mathbb{R}_+^0 \rightarrow \mathbb{R}_+^0$, positive on \mathbb{R}_+ and a positive constant K such that, for $|x| < K$,*

$$a(|x|) \leq V(t, x) \leq b(|x|)$$

holds. If there exists a continuous function $c : \mathbb{R}_+^0 \rightarrow \mathbb{R}_+^0$, positive on \mathbb{R}_+ such that

$$LV \leq -c(|x|)$$

holds, then the trivial solution ($X(t) = 0$) is globally asymptotically stable.

Recall that if $X(t; s, y)$ denotes the solution with initial condition $X(s) = y$ global asymptotic stability means that $\forall \epsilon > 0$ and $s \geq t_0$

$$\lim_{y \rightarrow 0} P\left(\sup_{t \geq t_0} |X(t; s, y)| \geq \epsilon\right) = 0$$

and

$$\lim_{y \rightarrow 0} P\left(\lim_{t \rightarrow +\infty} |X(t; s, y)| = 0\right) = 0.$$

We refer the reader to [7] and [15] for a complete study of these results.

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